Convergent normal forms of symmetric dynamical systems

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Abstract. It is shown that the presence of Lie point symmetries of (non-Hamiltonian) dynamical systems can ensure the convergence of the coordinate transformations which take the dynamical system (or vector field) into Poincaré–Dulac normal form.

1. Introduction

A well known and interesting procedure, going back to the classical work of Poincaré, for investigating analytic vector fields (VF) X_f

$$X_f \equiv \sum_{i=1}^{n} f_i(u)\frac{\partial}{\partial u_i} \equiv f \cdot \nabla \qquad (u \in \mathbb{R}^n) \tag{1}$$

or the associated dynamical systems (DS)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) \qquad u = u(t) \tag{2}$$

in a neighbourhood of a stationary point, is that of introducing some new coordinates in which the given VF takes its 'simplest' form, i.e. the normal form (NF) (in the sense of Poincaré–Dulac [1–4]). These coordinate transformations are usually performed by means of recursive techniques: in general, the normalizing transformations (NT) are actually purely formal transformations, and only very special conditions can ensure their convergence and the (local) analyticity of the NF [1–4]. In the investigation of these problems, a relevant role can be played by the presence of some symmetry property [5, 6] (see also [7]) of the VF X_f, i.e. by the presence of some VF X_g

$$X_g \equiv \sum_{i=1}^{n} g_i(u)\frac{\partial}{\partial u_i} \equiv g \cdot \nabla \tag{3}$$

such that

$$[X_f, X_g] = 0. \tag{4}$$

In terms of the DS (2), the symmetry VF X_g provides the Lie generator of a (possibly nonlinear) Lie point symmetry of the DS, and this can be conveniently expressed in the form of the Lie–Poisson bracket

$$\{f, g\}_i \equiv (f \cdot \nabla)g_i - (g \cdot \nabla)f_i = 0 \qquad (i = 1, \ldots, n) \tag{5}$$

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In this context, Bruno and Walcher [8] showed that the existence of an analytic symmetry for a two-dimensional DS is enough to ensure convergence of the NT; in [9] the convergence was also obtained for DS of dimension $n > 2$, combining the existence of symmetries with other conditions involving also the constants of motion of the DS; in [10, 11] the role of symmetries was investigated in view of the problem of linearizing the DS. In this paper, we will discuss some generalizations and results along the same lines.

2. Preliminary results

We will freely use f to both denote the VF X_f (1) and refer to (the rhs of) the DS (2); let us introduce the notation

$$f(u) = Au + F(u) \tag{6}$$

where f is assumed to be analytic in a neighbourhood of the stationary point $u_0 = 0$, and its linear part $A = (\nabla f)(0)$ a semisimple (and not zero) matrix. The NF of f will be written

$$\hat{f}(u) = Au + \hat{F}(u) \tag{7}$$

(the notation $\hat{\ }$ will be always reserved to NF; there is no danger of confusion if u is used to also denote the 'new' coordinates), and $\hat{F}(u)$ contains the 'resonant terms' with respect to A, i.e. the terms such that

$$\hat{F}(u) \in \mathrm{Ker}(\mathcal{A}) \tag{8}$$

where $\mathcal{A}$ is the 'homological operator' defined by

$$\mathcal{A}(h) = \{Au, h\}. \tag{9}$$

This fact can be conveniently stated in the following form.

Proposition 1. Every NF $\hat{f}$ admits the linear symmetry $g_A \equiv Au$.

Let us recall some well known and useful facts.

Lemma 1. Every symmetry

$$g(u) = Bu + G(u) \tag{10}$$

of a NF $\hat{f}$ is also a symmetry of the linear part Au of $\hat{f}$. The analogous result is true for the constants of motion of the DS (6): i.e. if a scalar function $\mu = \mu(u)$ is such that $\hat{f} \cdot \nabla\mu = 0$ then also $Au \cdot \nabla\mu = 0$.

Denoting by $\mathcal{G}_f$ and $\mathcal{I}_f$ the set of the symmetries and respectively of the constants of motion of f, we can then write

$$\mathcal{G}_{\hat{f}} \subseteq \mathcal{G}_{Au} \qquad \text{and} \qquad \mathcal{I}_{\hat{f}} \subseteq \mathcal{I}_{Au}. \tag{11}$$

This is true not only for analytic quantities, but also for quantities expressed by means of formal power series.

Lemma 2. Given the matrix A, the most general NF has the form

$$\hat{F}(u) = \sum_j \mu_j(u) M_j u \qquad \text{with } \mu_j(u) \in \mathcal{I}_{Au} \qquad \text{and} \qquad [M_j, A] = 0 \tag{12}$$

where the sum is extended to a set of linearly independent matrices M_j (the set of these matrices clearly includes A), and the constants of motion $\mu_j(u)$ appearing in (12) are expressed by (formal) power series and also possibly by quotients, i.e. by rational functions (see [12] for a detailed statement).

Lemma 3. If f admits a linear symmetry $g_B = Bu$, then the NF $\hat{f}$ also admits this symmetry. If f admits a (possibly formal) symmetry $g = Bu + G(u)$ and B is semisimple, then Bu is a symmetry of the NF $\hat{f}$, or—in other words—$\hat{F}$ is a NF also with respect to B, i.e. $\hat{F} \in \mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$.

The proofs of these lemmas are well known and can be found, for example, in [7, 12–14]. Let us also recall the basic conditions, in the form given by Bruno [1, 2], and called condition ω and condition A, which ensure the convergence of the NT of a given DS. Denoting by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix A, the first condition is (see [1–3] for details) as follows.

Condition ω. Let $\omega_k = \min |(q, \lambda) - \lambda_j|$ for all $j = 1, \ldots, n$ and all n-uple of nonnegative integers q_i such that $1 < \sum_{i=1}^{n} q_i < 2^k$ and $(q, \lambda) = \sum_i q_i \lambda_i \neq \lambda_j$: then

$$\sum_{k=1}^{\infty} 2^{-k} \ln(\omega_k^{-1}) < \infty.$$

This is a very weak condition, devised to control the appearance of small divisors in the series of NT, and generalizes the Siegel-type condition:

$$|(q, \lambda) - \lambda_j| > \epsilon \left(\sum_{i=1}^{n} |q_i| \right)^{-\nu}$$

for some $\epsilon, \nu > 0$, or the much simpler condition $|(q, \lambda) - \lambda_j| > \epsilon > 0$, for all n-uple q_i such that $(q, \lambda) \neq \lambda_j$ (see [1–3]). Hereafter we explicitly assume that this condition is always satisfied.

The other one, instead, is quite a strong restriction on the form of the NF. To state this condition in its simplest form, let us assume for a moment that there is a straight line through the origin in the complex plane which contains all the eigenvalues λ_i of A, and that there are eigenvalues lying on both parts of this line with respect to the origin. Then the condition reads as follows.

Condition A. There is a coordinate transformation changing f to $\hat{f}$, where $\hat{f}$ has the form

$$\hat{f} = Au + \alpha(u)Au$$

and $\alpha(u)$ is some scalar-valued power series (with $\alpha(0) = 0$).

In the case where no line in the complex plane satisfies the above property, then condition A should be modified [1] (or even weakened: for instance, if there is a straight line through the origin such that all the λ_i lie on the same side of this line, then the eigenvalues belong to a Poincaré domain [1, 3] and the convergence is guaranteed without any other condition); but in all the below applications, where in particular only linear NF will be ultimately concerned, the above formulation of condition A is enough to cover all the cases to be considered, and we can say [1, 2] that there is a convergent NT if the above conditions are satisfied. Clearly, here and in the following, 'convergence' means 'convergence in some open neighbourhood of $u_0 = 0$'.

3. Symmetries and convergence of the NT: A general result

Let us finally state the first result of this paper. It can be noted that quite strong assumptions are needed; but it is known, on the other hand, that the convergence of the NT is quite 'exceptional'. The examples given below will show how, thanks to additional symmetry properties, these assumptions can be verified. Let us remark that obviously, for any constant

c, then cf is a (trivial) symmetry of f; therefore, it is understood that when we assume the existence of some symmetry of f we will always refer to *nontrivial* symmetry, i.e. to symmetries $g \neq cf$.

Theorem 1. Given the analytic VF f, let us write its NF, according to lemma 2, in the form

$$\hat{f} = Au + \alpha(u)Au + \sideset{}{'}\sum \mu_j(u)M_j u \equiv Au + \alpha(u)Au + \hat{F}_1(u) \tag{13}$$

where (hereafter) $\sum'$ is the sum extended to the matrices $M_j \neq A$. Assume $\hat{F}_1(u) \neq 0$ (otherwise condition A is sufficient to ensure convergence of the NT), and:

(a) assume that f admits an analytic symmetry

$$g = Bu + G(u) \qquad \text{such that } B = aA \tag{14}$$

where a is a (possibly vanishing) constant;

(b) assume that the equation

$$\{\hat{F}_1, S\} = 0 \tag{15}$$

for the unknown

$$S = S(u) = \sideset{}{'}\sum v_j(u)M_j u \qquad \text{with } v_j(u) \in \mathcal{I}_{Au} \qquad \text{and} \qquad v_j(0) = 0 \tag{15'}$$

has only the trivial solution

$$S = c\hat{F}_1(u) \qquad (c = \text{constant}).$$

Then f can be put into NF by means of a convergent NT.

Proof. First, if $a = 0$ in assumption (a), one can consider, instead of g, the symmetry $g' = f + g$ having linear part Au; it is then not restrictive to assume $a = 1$, i.e. $B = A$. Once f is put into NF $\hat{f}$, the symmetry g will become a (possibly formal) symmetry $\widetilde{g}$

$$\widetilde{g} = Au + \beta(u)Au + \sideset{}{'}\sum v_j(u)M_j u \qquad \text{with } \beta(u), v_j(u) \in \mathcal{I}_{Au} \tag{16}$$

this is indeed the most general symmetry of a NF, thanks to lemma 1. The symmetry condition $\{f, g\} = 0$ in the new coordinates reads $\{\hat{f}, \widetilde{g}\} = 0$. Evaluating this bracket term by term, one is left with

$$\left\{ \alpha Au, \sideset{}{'}\sum_j v_j M_j u \right\} + \left\{ \sideset{}{'}\sum_j \mu_j M_j u, \beta Au \right\} + \left\{ \sideset{}{'}\sum_j \mu_j M_j u, \sideset{}{'}\sum_k v_k M_k u \right\} = 0$$

or

$$\left(\sideset{}{'}\sum_j \mu_j(u)M_j u \cdot \nabla\beta - \sideset{}{'}\sum_j v_j(u)M_j u \cdot \nabla\alpha \right) Au + \left\{ \sideset{}{'}\sum_j \mu_j(u)M_j u, \sideset{}{'}\sum_k v_k(u)M_k u \right\} = 0. \tag{17}$$

All other terms in fact vanish thanks to proposition 1 and lemmas 2 and 3. Now, in equation (17), the bracket $\{\cdot, \cdot\}$ produces, through the matrix commutators $[M_j, M_k]$, only terms proportional to $M_j u$ (and not to Au: this can easily be seen in a basis in which A is diagonal), therefore the terms appearing into the () and the bracket $\{\cdot, \cdot\}$ are both zero. The last bracket just has the form $\{\hat{F}_1, S\}$, and therefore assumption (b) gives $S = c\hat{F}_1$, i.e. $v_j(u) = c\mu_j(u)$. From the vanishing of the first () in (17), and using again assumption (b), one obtains similarly $\beta(u) = c\alpha(u)$, and then either $\widetilde{g} = c\hat{f}$, which is impossible because $g \neq cf$, or

$$\widetilde{g}(u) = Au. \tag{18}$$

This means that the transformation which puts $f(u)$ into $\hat{f}(u)$ transforms $g(u) = Au + G(u)$ into $\tilde{g}(u) = Au$, therefore the symmetry $g(u)$ satisfies condition A and there is convergent transformation which puts $g(u)$ into NF. Under this convergent transformation $f(u)$ is transformed into NF $\hat{f}$, as a consequence of the last part of lemma 3. □

Remark 1. One can see that assumption (b) of theorem 1 is equivalent to the assumption that the NF $\hat{f}$ admits only linear symmetries Lu. For a practical point of view (see the examples below), it is much simpler to verify property (b).

Remark 2. In the particular case where $\hat{F}_1(u)$ has the form

$$\hat{F}_1(u) = \mu(u)Mu \tag{19}$$

(with $M \neq A$), assumption (b) is actually equivalent to the very simple following one: there are no common (analytic, formal or fractional) constants of motion of the two linear problems

$$\dot{u} = Au \qquad \text{and} \qquad \dot{u} = Mu. \tag{20}$$

Indeed, assume there is some $\kappa = \kappa(u) \in \mathcal{I}_{Au} \cap \mathcal{I}_{Mu}$, then $S = \kappa(u)Au \neq c\hat{F}_1$ would satisfy $\{\hat{F}_1, S\} = 0$; notice, incidentally, that one would also get in this case $\kappa(u) \in \mathcal{I}_{\hat{f}}$. The converse is easily obtained by explicit calculations. This case has already been considered in [9]; the result for two-dimensional DS in [8] can be viewed as a particular case of this (see [9] for details).

The apparent difficulty in the application to concrete cases of the above results is that, in general, one does not known—*a priori*—the NF, and then it seems to be impossible to check if the assumptions of theorem 1 (or even condition A) are verified by the NF. However, as the foregoing examples will show, other symmetry properties of the VF may provide, once again, decisive help on this point. As already remarked, the convergence of the NT is to be considered a 'rare' event, and—as an obvious consequence—the examples are necessarily 'nongeneric'. On the other hand, the presence of linear or nonlinear symmetries of the equations has usually not only deep implications from the mathematical point of view (see [5–7]), but also a strong relevance for the interpretation in terms of the symmetry properties exhibited by the concrete physical problem. In particular, in the examples mentioned below, a relevant role is played by the rotation symmetry, namely a symmetry shared is a large class of physical problems (for instance, many problems and dynamical systems arising in fluid dynamics, see e.g. [14], and in several other fields, see e.g. [15]).

Example 1. Consider a three-dimensional analytic DS

$$\dot{u} = f(u) = Au + F(u) \qquad \text{with } A = \text{diag}(1, 1, -2) \tag{21}$$

and $u \equiv (x, y, z) \in \mathbb{R}^3$. Note first that condition ω is trivially satisfied, being $|(q, \lambda)|$ either equal to one of the eigenvalues $\lambda_j = 1, -2$ or $|(q, \lambda) - \lambda_j| \geqslant 1$, and $\omega_k = 1$ for all k. Assume that $f(u)$ possesses the linear SO_2 symmetry generated by $Lu \cdot \nabla$, where

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

i.e. f is 'equivariant' under rotations in the plane (x, y). Putting $r^2 = x^2 + y^2$, this implies that $F(u)$ must be written in the form

$$F(u) = \varphi_0(r^2, z)Au + \varphi_1(r^2, z)Iu + \varphi_2(r^2, z)Lu \tag{23}$$

where I is the identity matrix in $\mathbb{R}^3$. If the functions $\varphi_0, \varphi_1, \varphi_2$ have the following form

$$\varphi_0 = 0 \qquad \varphi_1 = a_1 r^2 z + a_2 z^3 + \text{higher-order terms} \qquad \varphi_2 = b\varphi_1 \tag{24}$$

where a_1, a_2, b are constants $\neq 0$, then the DS also admits the nonlinear symmetry

$$G(u) = r^2 z (I + bL) u. \tag{25}$$

Note that the assumption $a_2 \neq 0$ ensures that this DS is *not* a NF, and that the above symmetry is not trivial. Then assumption (a) of theorem 1 is satisfied. Now, the NF of the above DS (21)–(24) must be of the form

$$\hat{f} = Au + \alpha(r^2 z)Au + \mu_1(r^2 z)Iu + \mu_2(r^2 z)Lu \tag{26}$$

where α, μ_1, μ_2 depend only on $\kappa = r^2 z$, as a consequence of lemma 3 (i.e. the equivariance under SO_2 is preserved), and of Lemma 2 (the resonance condition). We have to look for the solutions S of the equation $\{\hat{F}_1, S\} = 0$, where the unknown S can be written

$$S = v_1(u)Iu + v_2(u)Lu \qquad \text{with } v_1(u), v_2(u) \in \mathcal{I}_{Au} \tag{27}$$

(it is easy to see that no other matrices can appear on the rhs of (26) and (27)), and where v_1, v_2 must be functions only of the two functionally independent quantities $x^2 z, xyz \in \mathcal{I}_{Au}$. The condition $\{\hat{F}_1, S\} = 0$ gives the first-order system of linear partial differential equations

$$(\mu_1 u + \mu_2 Lu) \cdot \nabla v_1 = v_1 u \cdot \nabla \mu_1$$
$$(\mu_1 u + \mu_2 Lu) \cdot \nabla v_2 = v_1 u \cdot \nabla \mu_2. \tag{28}$$

Observing that $Lu \cdot \nabla \mu_1 = 0$ and $Au \cdot \nabla \mu_1 = 0 = u \cdot \nabla \mu_1 - 3z \partial_z \mu_1$ and the same for v_1, i.e. $u \cdot \nabla v_1 = 3z \partial_z v_1$, one gets from the first of (28) the characteristic equation for the unknown $\zeta = \zeta(u)$ defined by $v_1(u) = \zeta(u)\mu_1(u)$

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{\mu_2}{\mu_1}\frac{dz}{3z} \tag{29}$$

which shows that ζ must be a function of r^2 and of some other variable of the form $v = \text{arctg}(y/x) + Z(z)$. On the other hand, μ_1 is a function of $\kappa = r^2 z$, and v_1 is a function of the quantities $x^2 z, xyz \in \mathcal{I}_{Au}$ (possibly also of x/y, of course, the only requirement is that μ_1, v_1 are power series in x, y, z); it is then easy to see that $\zeta = \text{constant}$, i.e. that $v_1 = c\mu_1$. Proceeding in the same way for the other equation in (28), one can conclude that

$$S = c\hat{F}_1$$

and then also assumption (b) in theorem 1 is satisfied, and therefore there is a convergent NT.

Note that it was essential in the calculations for the above example that μ_1 and μ_2 are both $\neq 0$, and this is in fact guaranteed by the normalizing procedure: indeed, at the lowest order, the resonant terms $r^2 z Iu$ and $r^2 z Lu$ are orthogonal to $z^3 Iu$ and $z^3 Lu$ (with respect to standard scalar product [3, 13, 14] introduced in the vector space of homogeneous polynomials, where the homological operator $\mathcal{A}$ is defined), then at the first step of the normalization procedure the resonant terms are not changed, i.e. one has $\mu_1 = a_1 r^2 z + \ldots$ and $\mu_2 = a_1 b\ r^2 z + \ldots$; and then—at any further step of the iteration—the lower-order terms are not altered.

4. Symmetries and convergence of the NT: A special case

Let us now return to the special case where $\hat{F}_1(u)$ can be written in the form

$$\hat{F}_1(u) = \mu(u)Mu. \tag{30}$$

In remark 2 we saw that assumption (b) of theorem 1 can be replaced by the requirement that there are no simultaneous constants of motion of Au and Mu. Assume now that, as

in example 1, the DS admits not only the nonlinear symmetry $g(u)$ (assumption (a) of theorem 1), but also a symmetry generated by some linear VF $Lu \cdot \nabla$, and that $g(u)$ satisfies

$$\{g, Lu\} = 0. \tag{31}$$

In the NT the symmetry is conserved step by step [14], therefore $g(u)$ also will be transformed, once f is in NF $\hat{f}$, into some $\tilde{g}(u)$ which is symmetric under Lu: $\{\tilde{g}, Lu\} = 0$. From this remark we see that it is sufficient to look for the common constants of motion of Au and Mu *only in the set* of those scalar functions $\kappa = \kappa(u)$ which are left invariant by Lu; in other words, we can conclude that if the set of these simultaneously invariant functions is trivial, i.e. it contains only constant numbers:

$$\mathcal{I}_{Au} \cap \mathcal{I}_{Mu} \cap \mathcal{I}_{Lu} = \mathbb{R} \tag{32}$$

then no other nonlinear symmetries are allowed, and $\tilde{g}(u)$ becomes necessarily $\tilde{g}(u) = Au$; then, using similar arguments as above, the convergence of the NT is guaranteed. In conclusion, we can state the following.

Theorem 2. Assume that f admits a symmetry g as in assumption (a) of theorem 1, and also a symmetry generated by a linear VF $Lu \cdot \nabla$ such that in addition (31) and (32) are satisfied. Then, if $\hat{F}_1$ has the form (30), the NT is convergent. The result can be trivially extended to the case that f admits an algebra of (more than one) symmetries $L_k u \cdot \nabla$.

Example 2. The same as example 1, here with $b = 0$ and $a_1, a_2 \neq 0$. In the NF now $\mu_1 = a_1 r^2 z + \ldots \neq 0$, but μ_2 may be zero. If $\mu_2 \neq 0$, theorem 1 can be applied. If $\mu_2 = 0$, then $\hat{F}_1$ has the form (30); on the other hand, it is clear that no SO_2-invariant analytic functions are simultaneously constants of motion of $\dot{u} = Au$ and $\dot{u} = u$, then all assumptions of theorem 2 are satisfied, and the NT is convergent.

The example given in [9] can be viewed as another example of theorem 2, in the presence of a larger symmetry (the Lie algebra of the group SO_3).

To conclude, it is interesting to point out the following peculiar property of the present approach.

Remark 3. All the results in this paper are peculiar of *non-Hamiltonian* DS: indeed, a Hamiltonian DS never satisfies the crucial hypothesis, i.e. assumption (b) of theorem 1. Let in fact $H = H(u)$, with $u \equiv (q, p) \in \mathbb{R}^{2m}$, be an analytic Hamiltonian and

$$\dot{u} = J \nabla H$$

be the associated DS, where J is the symplectic matrix. Writing $H = H_0 + H_R$, where H_0 is the quadratic part of H, we have clearly $Au = J \nabla H_0$ and $F(u) = J \nabla H_R$, and the requirement that $F(u)$ is in NF, $\hat{F}(u) = J \nabla \hat{H}_R$, now becomes the requirement that H_0 is a constant of motion of $\hat{H}_R$ (cf [16, 17]). Then, equation (15) always admits nontrivial solutions of the form

$$S = \eta(H_0) \hat{F}_1$$

for any (regular) function η of H_0.

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